**О. Мулява, М. Шеремета**

# КЛАСИ ЗБІЖНОСТІ АНАЛІТИЧНИХ ФУНКЦІЙ

**Монографія**

**O. Mulyava, M. Sheremeta**

# CONVERGENCE CLASSES OF ANALYTIC FUNCTIONS

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***Рецензенти:***
**Скасків О. Б.,** доктор фізико-математичних наук, професор, Львівський національний університет;
**Заболоцький М. В**., доктор фізико-математичних наук, професор, Львівський національний університет.

**Мулява О., Шеремета М.**
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Узагальнені класи збіжності є природними узагальненнями класичного класу збіжності, введеного Валіроном для вивчення властивостей цілих і мероморфних функцій. В запропонованій монографії ці класи застосовано до дослідження властивостей рядів Діріхле з додатними показниками, мероморфних функцій, характеристичних функцій ймовірнісних законів та адамарових композицій рядів.

Монографія буде корисною для спеціалістів, які працюють в теорії аналітичних функцій та її застосуваннях.

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***Reviewers***:
**Skaskiv Oleh**, Doctor of Physical and Mathematical Sciences, Professor, Lviv National University;
**Zabolotskyi Mykola**, Doctor of Physical and Mathematical Sciences, Professor, Lviv National University;

**O. Mulyava, M. Sheremeta**
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The generalized convergence classes are natural generalization of the classic convergence class, introduced by Valiron for the study of the entire and meromorphic function properties. In the proposed monograph these classes are applied to the study of properties of Dirichlet series with positive exponents, meromorphic functions, characteristic functions of probability laws and Hadamard compositions of series.

The monograph will be useful to the specialists working in the analytic functions theory and its applications.

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## CONTENTS

## PREFACE

One of major sections of modern complex analysis there is the theory of entire functions, founded as early as XIX century of by K. Weiersstrass and by J.Hadamard. In spite of its certain completeness many problems of this theory are unsolved until now, and at their decision there are new problems. The most widely-used descriptions of the growth of entire functions are an order and type, and in case of a zero type, belonging is used to the class of convergence. In the beginning the last century J. Valiron specified on necessary conditions of belonging of entire function to the convergence class in terms of its zeroes and coefficients of power development.

The growth of analytical in a unit disk functions in terms of order, type and class of convergence was investigated considerably later.

Direct generalization of power development of analytical function is a Dirichlet series with increasing to $+\infty$ exponents. For entire Dirichlet series with having a positive step exponents the analogue of theorem of J. Valiron got P. Kamthan. O. Mulyava removed the condition of positiveness of step in this statement, considering here belonging of entire and absolutely converging in a half-plane Dirichlet series to the generalized convergence classes.

Belonging to the classes of convergence is used in the theory of value distributions of entire and meromorphic functions, in the theory of characteristic functions of probability laws and in other spheres of complex analysis.

The material represented in this monograph is borrowed from the journal papers of the last years, that published mainly participants of Lviv school of mathematics.

We thank Yu.Trukhan for the help under the work on the monograph.

# Chapter 1

# DIRICHLET SERIES

A direct generalization of power development of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z = re^{i\theta}, \tag{1.1}$$

is a Dirichlet series

$$F(s) = a_0 + \sum_{n=1}^{\infty} a_n \exp\{s\lambda_n\}, \quad s = \sigma + it, \tag{1.2}$$

where $0 < \lambda_1 < \lambda_n \uparrow +\infty$.

Here we investigate in terms of convergence classes connections between the growth of the maximum modulus and the maximal term of Dirichlet series (1.2) and the behaviour its coefficients.

## 1.1 Preliminary information.

Let $\sigma_a$ be the abscissa of absolute convergence of series (1.2). It is known [22, p. 115] that if $\ln n = o(\lambda_n)$ as $n \to \infty$ then

$$\sigma_a = \alpha_0 := \varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}. \tag{1.3}$$

We remark that formula (1.3) also is true if $\dfrac{\ln(1/|a_n|)}{\ln n} \to +\infty$ as $n \to \infty$. Indeed, if $\alpha_0 < +\infty$ then for every $\alpha > \alpha_0$ there exists an increasing sequence

$(n_k)$ such that $|a_{n_k}| \geq \exp\{-\alpha\lambda_{n_k}\}$, that is $|a_{n_k}|\exp\{\sigma\lambda_{n_k}\} \geq 1$ for all $\sigma \geq \alpha_0$. Hence it follows that $\sigma_a \leq \alpha_0$. For $\alpha_0 = +\infty$ this inequality is trivial. On the contrary, if $\alpha_0 > -\infty$ then for every $\sigma < (1-\varepsilon)\alpha_0$, $0 < \varepsilon < 1$, we have

$$\ln|a_n| + \frac{\sigma\lambda_n}{1-\varepsilon} = \lambda_n\left(-\frac{1}{\lambda_n}\ln\frac{1}{|a_n|} + \frac{\sigma}{1-\varepsilon}\right) < 0$$

for all $n$ enough large. Therefore, for such $n$

$$|a_n|\exp\{\sigma\lambda_n\} = \exp\left\{-\varepsilon\ln\frac{1}{|a_n|} + (1-\varepsilon)\left(-\ln\frac{1}{|a_n|} + \frac{\sigma\lambda_n}{1-\varepsilon}\right)\right\} \leq$$

$$\leq \exp\left\{-\varepsilon\frac{\ln(1/|a_n|)}{\ln n}\ln n\right\},$$

whence it follows that for such $\sigma$ series (1.2) converges. Thus, $\sigma \leq \alpha_0$. For $\alpha_0 = -\infty$ this inequality is trivial.

By $S(\Lambda, A)$ we denote a class of Dirichlet series (1.2) with the sequence $\Lambda = (\lambda_n)$ of exponents and the abscissa $\sigma_a = A$ of absolute convergence. For $F \in S(\Lambda, A)$ and $\sigma < A$ we put $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ and let $\mu(\sigma, F) = \max\{|a_n|\exp\{\sigma\lambda_n\} : n \geq 0\}$ be the maximal term and $\nu(\sigma, F) = \max\{n : |a_n|\exp\{\sigma\lambda_n\} = \mu(\sigma, F)\}$ be its central index.

Then [22, p. 182] Cauchy inequality $\mu(\sigma, F) \leq M(\sigma, F)$ holds for all $\sigma < A$ and [22, p. 184]

$$\ln\mu(\sigma, F) = \ln\mu(\sigma_0, F) + \int_{\sigma_0}^{\sigma}\lambda_{\nu(t,F)}dt. \qquad (1.4)$$

Also we define

$$\varkappa_n(F) := \frac{\ln|a_n| - \ln|a_{n+1}|}{\lambda_{n+1} - \lambda_n}, \quad n \geq 0,$$

and will use properties of Newton's majorant for Dirichlet series (see [22, pp. 180-183] and [28]).

**Lemma 1.1.** *Let*

$$F_{mn}(s) = \sum_{n=}^{\infty} a_n^0 \exp\{s\lambda_n\} \qquad (1.5)$$

*be Newton's majorant for entire Dirichlet series from* $S(\Lambda, A)$. *Then* $\mu(\sigma, F_{mn}) = \mu(\sigma, F)$, $\nu(\sigma, F_{mn}) = \nu(\sigma, F)$, $\varkappa_n(F_{mn}) \nearrow A$ *as* $n \to \infty$ *and* $|a_n| \le a_n^0$ *for all* $n \ge 1$.

Let $p > 1$, $q = \dfrac{p}{p-1}$ and $f$ be a positive function on $(A, B)$, $-\infty \le A < B \le +\infty$. If $(\lambda_n^*)$ is a sequence of positive numbers and $(c_n)$ is a sequence of numbers from $(A, B)$ then

$$C_n := \frac{\lambda_1^* c_1 + \cdots + \lambda_n^* c_n}{\lambda_1^* + \cdots + \lambda_n^*} \in (A, B).$$

The following lemma generalizes a classical inequality of Hardy [17. p. 289].

**Lemma 1.2.** *If the function* $f^{1/p}$ *is convex on* $(A, B)$ *and the sequence* $(\mu_n)$ *is positive and non-increasing then for every* $\omega \le +\infty$

$$\sum_{n=1}^{\omega} \mu_n \lambda_n^* f(C_n) \le q^p \sum_{n=1}^{\omega} \mu_n \lambda_n^* f(c_n). \tag{1.6}$$

*Proof.* We put $t_n = \lambda_1^* + \cdots + \lambda_n^*$ for $n \ge 1$. Then

$$C_n = \frac{\lambda_1^* c_1 + \cdots + \lambda_{n-1}^* c_{n-1}}{t_n} + \frac{\lambda_n^* c_n}{t_n} = \frac{t_{n-1}}{t_n} C_{n-1} + \frac{\lambda_n^*}{t_n} c_n$$

and since $\dfrac{t_{n-1}}{t_n} + \dfrac{\lambda_n^*}{t_n} = 1$ and the function $f^{1/p}$ is convex, we have

$$f^{1/p}(C_n) \le \frac{t_{n-1}}{t_n} f^{1/p}(C_{n-1}) + \frac{\lambda_n^*}{t_n} f^{1/p}(c_n),$$

whence

$$-f^{1/p}(c_n) \le \frac{t_{n-1}}{\lambda_n^*} f^{1/p}(C_{n-1}) - \frac{t_n}{\lambda_n^*} f^{1/p}(C_n).$$

Therefore, in view of the equality $1/p + 1/q = 1$ we obtain

$$Q_n := \lambda_n^* f(C_n) - q\lambda_n^* f^{1/p}(c_n) f^{1/q}(C_n) \le$$

$$\le \lambda_n^* f(C_n) + q\lambda_n^* \left( \frac{t_{n-1}}{\lambda_n^*} f^{1/q}(C_n) f^{1/p}(C_{n-1}) - \frac{t_n}{\lambda_n^*} f^{1/q}(C_n) f^{1/p}(C_n) \right) =$$

$$= \lambda_n^* f(C_n) - q t_n f(C_n) + q t_{n-1} f^{1/q}(C_n) f^{1/p}(C_{n-1}) =$$

$$= (\lambda_n^* - q t_n) f(C_n) + q t_{n-1} f^{1/q}(C_n) f^{1/p}(C_{n-1}).$$

We consider on $[0, +\infty)$ a function $u(x) = \dfrac{x^p}{p} - ax + \dfrac{a^q}{q}$, where $a \geq 0$. This function has an unique point of the minimum $x = x(a) = a^{1/(p-1)}$ and $u(x(a)) = 0$. Hence it follows that $\dfrac{x^p}{p} + \dfrac{a^q}{q} \geq ax$ for all $a \geq 0$ and $x \geq 0$. Therefore, $Q_1 = -(q-1)\lambda_1^* f(C_1)$ and

$$Q_n \leq (\lambda_n^* - qt_n)f(C_n) + qt_{n-1}\left(\frac{f(C_n)}{q} + \frac{f(C_{n-1})}{p}\right) =$$

$$= (\lambda_n^* + t_{n-1} - qt_n)f(C_n) + \frac{q}{p}t_{n-1}f(C_{n-1}) =$$

$$= (1-q)t_n f(C_n) + \frac{q}{p}t_{n-1}f(C_{n-1}) = \frac{1}{p-1}(t_{n-1}f(C_{n-1}) - t_n f(C_n))$$

for $n \geq 2$ and, since the sequence $(\mu_n)$ non-increases, for $N < +\infty$ we have

$$\sum_{n=1}^{N} \mu_n Q_n \leq \frac{1}{p-1}\sum_{n=1}^{N} \mu_n(t_{n-1}f(C_{n-1}) - t_n f(C_n)) + \mu_1 Q_1 =$$

$$= \frac{1}{p-1}\sum_{n=1}^{N-1}(\mu_{n+1} - \mu_n)t_n f(C_n) - \mu_N t_N f(C_N) < 0.$$

Using the definition of $Q_n$ and the following inequality of Hölder

$$\sum_{n=1}^{N} a_n b_n \leq \left(\sum_{n=1}^{N} a_n^p\right)^{1/p} \left(\sum_{n=1}^{N} b_n^q\right)^{1/q},$$

hence we obtain

$$\sum_{n=1}^{N} \mu_n \lambda_n^* f(C_n) \leq q\sum_{n=1}^{N} \mu_n \lambda_n^* f^{1/p}(c_n) f^{1/q}(C_n) =$$

$$= q\sum_{n=1}^{N}(\mu_n \lambda_n^* f(c_n))^{1/p}(\mu_n \lambda_n^* f(C_n))^{1/q} \leq$$

$$\leq q\left(\sum_{n=1}^{N}(\mu_n \lambda_n^* f(c_n)\right)^{1/p} \left(\sum_{n=1}^{N}(\mu_n \lambda_n^* f(C_n)\right)^{1/q}.$$

Dividing this inequality into the last multiplier and involuting of $p$ we get (1.6) for $\omega = N$. In view of the arbitrariness of $N$ Lemma 1.2 is proved.

We need also the following statement.

**Lemma 1.3.** *Let* $\alpha : [1, +\infty) \to [0, +\infty)$ *and* $\gamma : [0, +\infty) \to [0, +\infty)$ *be nonnegative continuous increasing to* $+\infty$ *functions. If*

$$\overline{\lim_{n\to+\infty}} \frac{\alpha(n)}{\gamma(\lambda_n)} > 1, \tag{1.7}$$

*then there exists a subsequence* $(\lambda_k^*)$ *of the sequence* $(\lambda_n)$ *such that*

$$k \le \alpha^{-1}(\gamma(\lambda_k^*)) + 1 \tag{1.8}$$

*for all* $k \ge 1$ *and*

$$k_j \ge \alpha^{-1}(\gamma(\lambda_{k_j}^*)) \tag{1.9}$$

*for some increasing sequence* $(k_j)$ *of positive integers.*

*Proof.* In view of (1.7), the quantity $k_1 = \min\{k \ge 2 : \ k \ge \alpha^{-1}(\gamma(\lambda_k))\}$ is well defined. Clearly, $k < \alpha^{-1}(\gamma(\lambda_k))$ for $1 \le k \le k_1$ and

$$k_1 = k_1 - 1 + 1 < \alpha^{-1}(\gamma(\lambda_{k-1})) + 1 < \alpha^{-1}(\gamma(\lambda_k)) + 1.$$

Hence, if we set $\lambda_k^* = \lambda_k$ for $1 \le k \le k_1$ then (1.8) and (1.9) must hold for such $k$. Put now

$$j_1 = \min\{j \in \mathbb{N} : \ \ln(k_1 + 1) < \alpha^{-1}(\gamma(\lambda_{k_1+j}))\}$$

and from sequence $\Lambda = (\lambda_n)$ drop the members $\lambda_{k_1+1}, \dots, \lambda_{k_1+j_1}$, i. e. $\lambda_{k_1+1}^* = \lambda_{k_1+j_1}$. By virtue of (1.7), there exists

$$k_2 = \min\{k \ge k_1 + 1 : \ k \ge \alpha^{-1}(\gamma(\lambda_{k+j_1}))\}.$$

Then $k < \alpha^{-1}(\gamma(\lambda_{k+j_1}))$ for $k_1 + 1 \le k < k_2$, $k_2 \ge \alpha^{-1}(\gamma(\lambda_{k_2+j_1}))$ and

$$k_2 = k_2 - 1 + 1 \le \alpha^{-1}(\gamma(\lambda_{k_2+j_1})) + 1.$$

If we have already chosen $k_l$ and $j_{l-1}$, $l \ge 2$, then we put

$$j_l = \min\{j \in \mathbb{N} : \ \ln(k_l + 1) < \alpha^{-1}(\gamma(\lambda_{k_1+j_1+\dots+j_{l-1}+j}))\}$$

and delete the terms

$$\lambda_{k_1+j_1+\dots+j_{l-1}+1}, \dots, \lambda_{k_1+j_1+\dots+j_{l-1}+j_l}$$

from $\Lambda$, i. e. we set

$$\lambda^*_{k_{l+1}} = \lambda_{k_1+j_1+\dots+j_{l-1}+j_l}.$$

As before, in view of (1.7), the quantity

$$k_{l+1} = \min\{k \geq k_l + 2 : \ k \geq \alpha^{-1}(\gamma(\lambda_{k+j_1+\dots+j_{l-1}+j_l}))$$

is well defined. Then

$$k < \alpha^{-1}(\gamma(\lambda_{k+j_1+\dots+j_l})), \quad k_l + 1 \leq k < k_{l+1},$$

$$k_{l+1} \geq \alpha^{-1}(\gamma(\lambda_{k_{l+1}+j_1+\dots+j_l}))$$

and, as before,

$$k_{l+1} \leq \alpha^{-1}(\gamma(\lambda_{k_{l+1}+j_1+\dots+j_l})) + 1.$$

Hence, if $\lambda^*_k = \lambda_{k+j_1+\dots+j_{l-1}+j_l}$ for $k_l + 1 \leq k \leq k_{l+1}$ then we again have (1.8) and (1.9) for such $k$. Since $l$ was chosen arbitrary, Lemma 1.3 is proved.

## 1.2 Belonging of entire Dirichlet series to convergence classes

J. Valiron [71, p.18] proved, that if an entire function (1.1) has the order $\varrho \in (0, +\infty)$ and belongs to the convergence class, i. e.

$$\int\limits_1^\infty \frac{\ln M_f(r)}{r^{\varrho+1}} < +\infty, \quad M_f(r) = \max\{|f(z)| : |z| = r\},$$

then

$$\sum_{n=1}^\infty |a_n|^{\varrho/n} < +\infty.$$

P. Kamthan [19] generalized this result on the case of entire (absolutely convergent in $\mathbb{C}$) Dirichlet series (1.2). He showed that if the sequence $(\lambda_n)$ has a positive finite step, that is $0 < h \leq \lambda_{n+1} - \lambda_n \leq H < \infty$ for $n \geq 0$ and $\varkappa_n(F) \uparrow +\infty$ as $n \to \infty$ then in order that

$$\int\limits_0^\infty \frac{\ln M(\sigma, F)}{\exp\{\varrho\sigma\}} < +\infty, \tag{1.10}$$

it is necessary and sufficient that

$$\sum_{n=1}^{\infty} |a_n|^{\varrho/\lambda_n} < +\infty.$$

Giving up a positive and finite step of $(\lambda_n)$, in the article [28] a result is got considerably stronger. It is proved that if entire Dirichlet series (1.2) has $R$-order $\varrho_R = \varrho \in (0, +\infty)$ and $\ln n = O(\lambda_n)$ as $n \to \infty$ then in order that correlation (1.3) holds it is necessary and in the case, when the sequence $(\varkappa_n(F))$ non-decreases, it is sufficient that

$$\sum_{n=1}^{\infty} (\lambda_n - \lambda_{n-1})|a_n|^{\varrho/\lambda_n} < +\infty.$$

Except this statement in the article [29] the results are given about belonging of Dirichlet series to the generalized convergence class.

To give this definition we denote, as in [58], by $L$ a class of continuous nonnegative on $(-\infty, +\infty)$ functions $\alpha$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each fixed $c \in (0, +\infty)$, i. e. $\alpha$ is slowly increasing function. Clearly, $L_{si} \subset L^0$.

We need the following properties of the functions from $L^0$.

**Lemma 1.4.** *Let* $\alpha \in L$ *and* $A(\delta) = \varlimsup\limits_{x\to+\infty} \dfrac{\alpha((1+\delta)x)}{\alpha(x)}$, $\delta > 0$. *In order that* $\alpha \in L^0$, *it is necessary and sufficient that* $A(\delta) \to 1$ *as* $\delta \to 0$.

*If* $\alpha \in L^0$ *then* $\alpha$ *is* $RO$*-increasing [65], i. e. for every* $h \in [1, a]$, $1 < a < +\infty$, *and all* $x \geq x_0$ *the inequality* $\dfrac{\alpha(hx)}{\alpha(x)} \leq M(a) < +\infty$ *is true.*

*Proof.* Suppose that $\alpha \in L^0$ and $A(\delta) \not\to 1$ as $\delta \to 0$. Since the function $A(\delta)$ is non-decreasing, there exists $\lim\limits_{\delta\downarrow 0} A(\delta) = a^* > 1$, i. e. $A(\delta) \geq a^* > 1$. We choose an arbitrary sequence $(\delta_n) \downarrow 0$. For each $(\delta_n)$ there exists a sequence $(x_{n,k})$ such that $(x_{n,k}) \uparrow +\infty$ as $k \to \infty$ and $\alpha((1 + \delta_n)x_{n,k}) \geq$ $\geq a\alpha(x_{n,k})$, $1 < a < a^*$. We put

$$x_1 = x_{1,1}, \quad x_n = \min\{x_{n,k} : x_{n,k} \geq x_{n-1}, k \geq n - 1\}$$

and construct a function $\gamma(x) \to 0$ $(x \to +\infty)$ such that $\gamma(x_n) = \delta_n$. Then

$$\alpha((1+\gamma(x_n))x_n) = \alpha((1+\delta_n))x_n) \geq a\alpha(\delta_n).$$

In view of the definition of $L^0$ this is impossible.

On the contrary, let $A(\delta) \to 1$ as $\delta \to 0$ and $\alpha \notin L^0$. Then there exist a function $\gamma(x) \to 0$ $(x \to +\infty)$ and a sequence $(x_k) \uparrow +\infty$ such that

$$\lim_{k\to\infty} \frac{\alpha((1+\gamma(x_k))x_k)}{\alpha(x_k)} = a \neq 1.$$

Clearly, $a < 1$ provided $\gamma(x_k) < 0$ and $a > 1$ provided $\gamma(x_k) > 0$. We examine, for example, the second case. Let $\delta > 0$. Then $\gamma(x_k) < \delta$ for $k \geq k_0$ and

$$\overline{\lim_{x\to+\infty}} \frac{\alpha((1+\delta)x)}{\alpha(x)} \geq \overline{\lim_{k\to\infty}} \frac{\alpha((1+\delta)x_k)}{\alpha(x_k)} \geq \overline{\lim_{k\to+\infty}} \frac{\alpha((1+\gamma(x_k))x_k)}{\alpha(x_k)} = a,$$

that is $A(\delta) \geq a$ and $\lim\limits_{\delta\downarrow 0} A(\delta) > 1$, which is impossible.

The first part of Lemma 1.4 is proved.

Now we prove second part. Since the function $\alpha \in L^0$ is increasing, it is sufficient to prove that

$$c(2) := \overline{\lim_{x\to+\infty}} \frac{\alpha(2x)}{\alpha(x)} < +\infty.$$

We suppose, on the contrary, that $c(2) = +\infty$, i. e. there exists a sequence $(x_k) \uparrow +\infty$ such that

$$\frac{\alpha(2x_k)}{\alpha(x_k)} = w(x_k) \to +\infty, \quad k \to \infty.$$

We may assume that $2x_k < x_{k+1}$ for $k \geq 1$ and $w(x_k)^{1/k} \to +\infty$ as $k \to \infty$.

We divide the interval $[x_k, 2x_k]$ into $k$ equal parts by the points $x_k^{(j)} = x_k + \dfrac{j}{k} x_k$, $0 \leq j \leq k$. Then there exists $j_k$, $0 \leq j_k \leq k-1$, such that $\dfrac{\alpha(x_k^{(j_k+1)})}{\alpha(x_k^{(j_k)})} \geq w(x_k)^{1/k}$, because if $\dfrac{\alpha(x_k^{(j+1)})}{\alpha(x_k^{(j)})} < w(x_k)^{1/k}$ for all $j$, $0 \leq j \leq k-1$, then

$$w(x_k) = \frac{\alpha(2x_k)}{\alpha(x_k)} = \frac{\alpha(x_k^{(k)})}{\alpha(x_k^{(k-1)})} \cdots = \frac{\alpha(x_k^{(1)})}{\alpha(x_k^{(0)})} < (w(x_k)^{1/k})^k = w(x_k).$$

Thus, $\dfrac{\alpha(x_k^{(j_k+1)})}{\alpha(x_k^{(j_k)})} \to +\infty$ and $\dfrac{x_k^{(j_k+1)}}{x_k^{(j_k)}} = 1 + \dfrac{1}{k+j_k} \to 1$ as $k \to \infty$. Hence it follows that $\alpha \notin L^0$, which is impossible. Therefore, $c(2) < +\infty$, i. e. $\alpha$ is $RO$-increasing. The proof of Lemma 1.4 is complete.

In [29] the following theorem was proved.

**Theorem 1.1.** *Let $\alpha$ be a concave function on $[x_0, +\infty)$ and $\alpha(e^x) \in L^0$, and a function $\beta \in L^0$ satisfies the conditions $x\beta'(x)/\beta(x) \ge h > 0$ for $x \ge x_0$ and*

$$\int_{x_0}^{\infty} \frac{\alpha(x)}{\beta(x)} dx < +\infty. \tag{1.11}$$

*Suppose that the exponents $\lambda_n$ of entire Dirichlet series* (1.2) *satisfies the condition $\ln n = o(\lambda_n \beta^{-1}(\alpha(\lambda_n)))$ as $n \to \infty$. Then in order that*

$$\int_{\sigma_0}^{\infty} \frac{\alpha(\ln M(\sigma, F))}{\beta(\sigma)} d\sigma < +\infty, \tag{1.12}$$

*it is necessary and in the case, when the sequence $(\varkappa_n(F))$ non-decreases, it is sufficient that*

$$\sum_{n=1}^{\infty} (\alpha(\lambda_n) - \alpha(\lambda_{n-1}))\beta_1\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right) < +\infty, \quad \beta_1(x) = \int_x^{\infty} \frac{dt}{\beta(t)}.$$

For $\alpha \in L$ and $\beta \in L$, except the generalized convergence $\alpha\beta$-class defined by condition (1.12), for entire Dirichlet series we define also a modified generalized convergence $\alpha\beta$-class by condition

$$\int_{\sigma_0}^{\infty} \frac{1}{\beta(\sigma)} \alpha\left(\frac{\ln M(\sigma, F)}{\sigma}\right) d\sigma < +\infty. \tag{1.13}$$

Here we obtain an analog of Theorem 1.1 for the modified generalized convergence $\alpha\beta$-class.

**1.2.1. Modified generalized convergence $\alpha\beta$-class in terms of maximal term.** We investigate conditions under which correlation (1.13) is equivalent to the correlation

$$\int_{\sigma_0}^{\infty} \frac{1}{\beta(\sigma)} \alpha\left(\frac{\ln \mu(\sigma, F)}{\sigma}\right) d\sigma < +\infty. \tag{1.14}$$

In view of Cauchy inequality (1.13) implies (1.14). For the proof of the converse we need the following lemma.

**Lemma 1.5.** *If for entire Dirichlet series* (1.2)

$$h_0 := \overline{\lim_{n\to\infty}} \frac{\ln n}{-\ln |a_n|} < 1$$

*then for every* $\epsilon \in (0, 1-h_0)$ *there exists* $A_0(\epsilon) > 0$ *such that for all* $\sigma \geq 0$

$$M(\sigma, F) \leq A_0(\epsilon) \mu\left(\frac{\sigma}{1-h_0-\epsilon}, F\right).$$

*Proof.* For $n \geq 1$ we put $r_n = \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}$. Then $r_n \to +\infty$ as $n \to \infty$ and

$$M(\sigma, F) \leq |a_0| + \left(\sum_{r_n \leq \sigma/(1-h_0-\epsilon)} + \sum_{r_n > \sigma/(1-h_0-\epsilon)}\right) |a_n| \exp\{\sigma\lambda_n\} =$$

$$= |a_0| + \sum_{r_n \leq \sigma/(1-h_0-\epsilon)} |a_n| \exp\left\{\frac{\sigma\lambda_n}{1-h_0-\epsilon}\right\} \exp\left\{-\frac{(h_0+\epsilon)\sigma\lambda_n}{1-h_0-\epsilon}\right\} +$$

$$+ \sum_{r_n > \sigma/(1-h_0-\epsilon)} |a_n| \exp\{\sigma\lambda_n\} \leq$$

$$\leq |a_0| + \mu\left(\frac{\sigma}{1-h_0-\epsilon}, F\right) \sum_{r_n \leq \sigma/(1-h_0-\epsilon)} \exp\{-(h_0+\epsilon)\lambda_n r_n\} +$$

$$+ \sum_{r_n > \sigma/(1-h_0-\epsilon)} |a_n| \exp\{(1-h_0-\epsilon) r_n \lambda_n\} \leq$$

$$\leq |a_0| + \left(\mu\left(\frac{\sigma}{1-h_0-\epsilon}, F\right) + 1\right) \sum_{n=1}^{\infty} \exp\{(h_0+\epsilon) \ln |a_n|\}.$$

Since in view of the definition of $h_0$ we have $(h_0+\epsilon/2) \ln |a_n| \leq -\ln n$ for all $n$ enough large, hence we obtain necessary inequality. Lemma 1.5 is proved.